Deptt- MATHEMATICS

College- SOGHRA COLLEGE,BIHAR SHARIF

Part- BSc PART 2

6. SOLUTIONS OF DIFFERENTIAL EQUATIONS

Finding the dependent variable from the differential equation is called solving or integrating it. The solution or the integral of a differential equation is, therefore, a relation between the dependent and independent variables (free from derivatives) such that it satisfies the given differential equation.

Note: The solution of the differential equation is also called its primitive.

There can be two types of solution to a differential equation:

(a) General solution (or complete integral or complete primitive)

A relation in x and y satisfying a given differential equation and involving exactly the same number of arbitrary constants as the order of the differential equation.

(b) Particular solution

A solution obtained by assigning values to one or more than one arbitrary constant of general solution

Illustration 8: The general solution of $x^2\frac{dy}{dx} = 2$ is

Sol: First separate out x term and y term and then integrate it, we shall obtain result.

$\frac{dy}{dx} = \frac{2}{x^2} \Rightarrow dy = \frac{2}{x^2}dx$ Now integrate it. We get $y = -\frac{2}{x} + c$

Illustration 9: Verify that the function $x + y = \tan^{-1}y$ is a solution of the differential equation $y^2y' + y^2 + 1 = 0$

Sol: By differentiating the equation $x + y = \tan^{-1}y$ with respect to x we can prove the given equation.

We have, $x + y = \tan^{-1}y$ ... (i)

Differentiating (i), w.r.t. x we get

$$1 + \frac{dy}{dx} = \frac{1}{1+y^2}\frac{dy}{dx} \Rightarrow 1 + \frac{dy}{dx}\left(\frac{1+y^2-1}{1+y^2}\right) = 0$$

$$\Rightarrow (1 + y^2) + y^2\frac{dy}{dx} = 0 \Rightarrow y^2y' + y^2 + 1 = 0$$

Illustration 10: Show that the function $y = Ax + \left(2x+2y\frac{dy}{dx}\right)$ is a solution of the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

Sol: Differentiating $y = Ax + \frac{B}{x}$ twice with respect to x and eliminating the constant term, we can prove the given equation.

We have, $y = Ax + \frac{dy}{dx} \Rightarrow xy = Ax^2 + B$ ... (i)

Differentiation (i) w.r.t. 'x'. we get $\Rightarrow x\frac{dy}{dx} + 1.y = 2Ax$ ... (ii)

Again differentiating (ii) w.r.t., 'x', we get

$$\Rightarrow x.\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 2A \qquad \Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \frac{x\frac{dy}{dx}+y}{x} \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

Which is same as the given differential equation. Therefore $y = Ax + \frac{dy}{dx}$ is a solution for the given differential equation.

Illustration 11: If $y.\sqrt{x^2+1} = \log\left[\sqrt{x^2+1}\right]$ show that $(x^2 + 1)\frac{dy}{dx} + xy + 1 = 0$

Sol: Similar to the problem above, by differentiating $y.\sqrt{x^2+1} = \log\left[\sqrt{x^2+1} - x\right]$ one time with respect to x, we will prove the given equation.

We have, $y.\ \sqrt{x^2+1} = \log\left[\sqrt{x^2+1}\right]$ ... (i)

Differentiating (i), we get

$$\sqrt{x^2+1}\frac{dy}{dx} + \frac{1}{2}\frac{2x}{\sqrt{x^2+1}}y = \frac{(1/2)\left(2x/\sqrt{x^2+1}\right)-1}{\sqrt{x^2+1}-x} \Rightarrow \sqrt{x^2+1}\frac{dy}{dx} + \frac{x}{\sqrt{x^2+1}} = \frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}\left[\sqrt{x^2+1}-x\right]};$$

$$(x^2+1)\frac{dy}{dx} + xy = \frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}-x}; \qquad (x^2 + 1)\frac{dy}{dx} + xy = -1; \qquad (x^2 + 1)\frac{dy}{dx} + xy + 1 = 0$$

Illustration 12: Show that y = acos(logx) + bsin(logx) is a solution of the differential equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Sol: As the given equation has two arbitrary constants, hence differentiating it two times we can prove it.

We have, y = acos(logx) + bsin(logx) ... (i)

Differentiating (i) w.r.t 'x'. we get ; $\frac{dy}{dx} = -\frac{a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$

$$x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x) \qquad \text{... (ii)}$$

Again differentiating with respect to 'x', we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -[a\cos(\log x) + b\sin(\log x)] \qquad \Rightarrow \frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \qquad \Rightarrow \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Which is same as the given differential equation

Hence, y = acos(logx) + bsin(logx) is a solution of the given differential equation.

7. METHODS OF SOLVING FIRST ORDER FIRST DEGREE DIFFERENTIAL EQUATION

7.1 Equation of the Form dy/dx = f(x)

To solve this type of differential equations, we integrate both sides to obtain the general solution as discussed below

$$\frac{dy}{dx} = f(x) \quad \Rightarrow \quad dy = f(x)dx$$

Integrating both sides we obtain $\int dy = \int f(x)dx + c \Rightarrow y = \int f(x)dx + c$

Illustration 13: The general solution of the differential equation $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}$ is

Sol: General solution of any differential equation is obtained by integrating it hence for given equation we have to integrate it one time to obtain its general equation.

We have: $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}$

Integrating, $y = \int\left(x^5 + x^2 - \frac{2}{x}\right) dx + c = \int x^5 dx + \int x^2 dx - 2\int \frac{1}{x} dx + c \Rightarrow y = \frac{x^6}{6} + \frac{x_3}{3} - 2\log|x| + c$

Which is the required general solution.

Illustration 14: The solution of the differential equation $\cos^2 x \frac{d^2y}{dx^2} = 1$ is

Sol: By integrating it two times we will get the result.

$\cos^2 x \frac{d^2y}{dx^2} = 1 \Rightarrow \frac{d^2y}{dx^2} = \sec^2 x$

On integrating, we get $\frac{dy}{dx} = \tan x + c_1$

Integrating again, we get $y = \log(\sec x) + c_1 x + c_2$

7.2 Equation of the form dy/dx = f(x) g(y)

To solve this type of differential equation we integrate both sides to obtain the general solution as discussed below

$\frac{dy}{dx} = f(x)g(y) \Rightarrow g(y)^{-1} dy = f(x)dx$

Integrating both sides, we get $\int (g(y))^{-1} dy = \int f(x)dx$

Illustration 15: The solution of the differential equation log(dy/dx) = ax + by is

Sol: We can also write the given equation as $\frac{dy}{dx} = e^{ax+by}$. After that by separating the x and y terms and integrating both sides we can get the general equation.

$\frac{dy}{dx} = e^{ax+by} \Rightarrow \frac{dy}{dx} = e^{ax+by} \Rightarrow e^{-by} dy = e^{ax} dx \Rightarrow -\frac{1}{b}e^{-by} = \frac{1}{a}e^{ax} + c$

Illustration 16: The solution of the differential equation $\frac{dy}{dx} = e^{x+y} + x^2 e^y$ is

Sol: Here first we have to separate the x and y terms and then by integrating them we can solve the problem above.

The given equation is $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

$\Rightarrow \frac{dy}{dx} = e^x.e^y + x^2 e^y \Rightarrow e^{-y} dy = (e^x + x^2)dx$, Integrating, $\int e^{-y} dy = \int\left(e^x + x^2\right)dx + c$

$\Rightarrow \frac{e^{-y}}{-1} + e^x + \frac{x^3}{3} + c \quad \Rightarrow -\frac{1}{e^y} = e^x + \frac{1}{3}x^3 + c \Rightarrow e^x + \frac{1}{e^y} + \frac{x^3}{3} = C$

7.3 Equation of the Form dy/dx = f (ax+by+c)

To solve this type of differential equation, we put ax + by + c = v and $\frac{dy}{dx}=\frac{1}{b}\left(\frac{dy}{dx}-0\right)$

$\therefore\ \frac{dy}{a+bf(v)}$ = dx

So solution is by integrating $\int\frac{dy}{a+bf(v)}=\int dx$

Illustration 17: $(x+y)^2\frac{dy}{dx}=a^2$

Sol: Here we can't separate the x and y terms, therefore put x + y = t hence $\frac{dy}{dx}=\frac{dt}{dx}-1$. Now we can easily separate the terms and by integrating we will get the required result.

Let $x + y = t \Rightarrow t^2\left(\frac{dt}{dx}-1\right) = a^2$; $\frac{dt}{dx}=\frac{a^2}{t^2}+1 = \frac{a^2+t^2}{t^2} \Rightarrow \int\frac{t^2dt}{t^2+a^2} = x + c$

$\Rightarrow\quad t-a\tan^{-1}\frac{dy}{dx} = x + c \Rightarrow\quad y-a\tan^{-1}\frac{x+y}{a}=c$

Illustration 18: $\frac{dx}{dx} = \frac{x+y-1}{\sqrt{x+y+1}}$

Sol: Put x + y +1 = t^2 and then solve similar to the above illustration.

let x + y +1 = t^2

$\Rightarrow\quad \left(2t\frac{dt}{dx}-1\right) = \frac{t^2-2}{t} \quad\Rightarrow\quad \frac{2tdt}{dx}=\frac{t^2+t-2}{t} \quad\Rightarrow\quad \int\frac{2t^2}{(t-1)(t+2)}dt = x+c$

$\Rightarrow\quad 2\int\left(1+\frac{1}{3(t-1)}-\frac{4}{3(t+2)}\right)dt = x+c \quad\Rightarrow\quad 2t+\frac{2\ln|t-1|}{3}-\frac{8\ln|t+2|}{3} = x+c$

$\Rightarrow\quad 2\sqrt{x+y+1}+\frac{2\ln|\sqrt{x+y+1}-1|}{3}-\frac{8\ln|\sqrt{x+y+1}+2|}{3} = x+c$

Illustration 19: $\frac{dy}{dx}$ = cos(10x + 8y). Find curve passing through origin in the form y = f(x) satisfying differential equations given

Sol: Here first put 10x + 8y = t and then taking integration on both sides we will get the required result.

Let 10x + 8y = t

$\Rightarrow\quad 10+8\frac{dy}{dx}=\frac{dt}{dx} \quad\Rightarrow\quad \frac{dy}{dx}-10 = 8\cos t \quad\Rightarrow\quad \int\frac{dt}{8\cos t+10}\int dx = x + c$

$p=\tan t/2 \quad \frac{dp}{dx}=\frac{1+p^2}{2(1)}\frac{dy}{dx} \Rightarrow \frac{dt}{dx}=\frac{2dp}{1+p^2}$

$\therefore\int\frac{\frac{2dp}{1+p^2}}{8\left(\frac{1-p^2}{1+p^2}\right)+10} = \int\frac{2dp}{1p^2+18} = \int\frac{dp}{p^2+9} = x+c$

$\Rightarrow\ \tan^{-1}(P/3) = x+c \ \Rightarrow\ \tan^{-1}\left(\frac{\tan(t/2)}{3}\right) = x+c \Rightarrow\ 3\tan(x+c) = \tan(10x+84)$

7.4 Parametric Form

Some differential equations can be solved using parametric forms.

Case I:

$x = r\cos\theta$ $y = r\sin\theta$

Squaring and adding $x^2 + y^2 = r^2$ ... (i)

$\tan\theta = \int e^{-y}dy = \int (e^x + x^2)dx + c$ ... (ii)

$xdx + ydy = rdr$ ... (iii)

$\sec^2\theta\, d\theta = \frac{e^{-y}}{-1} = e^x + \frac{x^3}{3} + c$ $\Rightarrow xdy - ydx = x^2\sec^2\theta\, d\theta$ $x = r\cos\theta;\ xdy - ydx = r^2 d\theta$

Case II:

If $x = r\sec\theta$, $y = r\tan\theta$

$x^2 - y^2 = r^2$ ... (i)

$\frac{1}{e^y} = e^x + \frac{1}{3}x^3 + c = \sin\theta$ ... (ii)

$\Rightarrow$ $xdx - ydy = rdr;$ $xdy - ydx = \cos\theta\, x^2\, d\theta$ $\Rightarrow$ $xdy - ydx = r^2\sec\theta d\theta$

Illustration 20: Solve $xdx + ydy = x(xdy - ydx)$

Sol: By substituting $x = r\cos\theta$ and $y = r\sin\theta$ the given equation reduces to $rdr = r\cos\theta(r^2d\theta)$. Hence by separating and integrating both sides we will get the result.

Let $x = r\cos\theta,\ y = r\sin\theta$

Hence the given equation becomes $rdr = r\cos\theta(r^2d\theta)$

$$\int\frac{dr}{r^2} = \int\cos\theta d\theta \quad\Rightarrow\quad -\frac{1}{r} = \sin\theta + c \quad\Rightarrow\quad -\frac{1}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + c$$

Illustration 21: Solve $\dfrac{x + y\frac{dy}{dx}}{x\frac{dy}{dx} - y} = \sqrt{\dfrac{1-x^2-y^2}{x^2+y^2}}$

Sol: Similar to the problem above, by substituting $x = r\cos\theta$ and $y = r\sin\theta$ the given equation reduces to $\frac{rdr}{r^2d\theta} = \frac{\sqrt{1-r^2}}{r}$. Hence by integrating both sides we will get the result.

$$\frac{x + y\frac{dy}{dx}}{x\frac{dy}{dx} - y} = \sqrt{\frac{1-x^2-y^2}{x^2+y^2}} \quad\Rightarrow\quad \frac{xdx+ydy}{xdy-ydx} = \sqrt{\frac{1-x^2-y^2}{x^2+y^2}}$$

Let $x = r\cos\theta,\ y = r\sin\theta$

$$\frac{rdr}{r^2d\theta} = \frac{\sqrt{1-r^2}}{r} \quad\Rightarrow\quad \int\frac{dr}{\sqrt{1-r^2}} = \theta + c \quad\Rightarrow\quad \sin^{-1}r = \theta + c$$

$$\Rightarrow\quad \sin^{-1}\sqrt{x^2+y^2} = \sin^{-1}\frac{y}{\sqrt{x^2+y^2}} + c$$

Illustration 22: $\frac{xdx + ydy}{\sqrt{x^2+y^2}} = \frac{ydx - xdy}{x}$

Sol: Similar to the above illustration.

Let $x = r\cos\theta,\ y = r\sin\theta$

$\Rightarrow\ -\frac{rdr}{r^2 d\theta} = \frac{\sqrt{r^2}}{r\cos\theta}$ $\Rightarrow\ \int \sec\theta d\theta + \int \frac{dr}{r} = 0$

$\Rightarrow\ \log(\sec\theta + \tan\theta) + \log r = c$ $\Rightarrow\ x^2 + y^2 + y\left(\sqrt{x^2+y^2}\right) + Cx = 0$

7.5 Homogeneous Differential Equations

A differential equation in x and y is said to be homogeneous if it can be put in the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, where f(x, y) and g(x, y) are both homogeneous function of the same degree in x and y.

To solve the homogeneous differential equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$,

substitute y = vx and so $\frac{dy}{dx} = v + x\frac{dy}{dx}$

Thus differential reduces to the form $v + x\frac{dy}{dx} = f(v) \Rightarrow \frac{dx}{x} = \frac{dv}{f(v) - v}$

Therefore, solution is $\int \frac{dy}{x} = \int \frac{dv}{f(v) - v} + c$

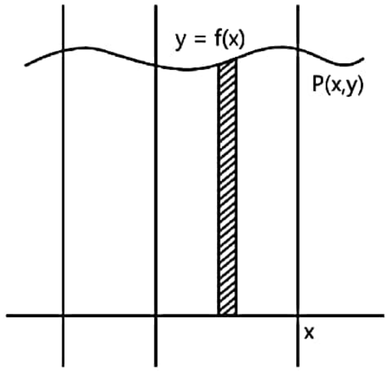

Figure 24.1

Illustration 23: Find the curve passing through (1, 0) such that the area bounded by the curve, x-axis and 2 ordinates, one of which is constant and other is variable, is equal to the ratio of the cube of variable ordinate to variable abscissa.

Sol: By differentiating $\int_c^x ydx = \frac{y^3}{x}$, we will get the differential equation.

$A = \int_c^x ydx = \frac{y^3}{x}$ $\Rightarrow y = \frac{x.3y^2y' - y^3.1}{x^2}$ $\Rightarrow x^2 = 3xyy' - y^2$ $\Rightarrow\ \frac{dy}{dx} = \frac{x^2+y^2}{3xy}$

(On differentiating the first integral equation w.r.t x)

Put y = vx; $v + x\frac{dt}{dx} = \frac{1+v^2}{3v}$ $\Rightarrow\ \int \frac{3v}{1-2v^2}dv = \int \frac{1}{x}dx$ $\Rightarrow\ -\frac{3}{4}\log\left|1-2v^2\right| = \log x + \log c$ $\Rightarrow\ (x^2 - 2y^2)^3 = cx^2$

Given this curve passes through (1, 0). So, c=1 Hence the equation of curve is $(x^2 - 2y^2)^3 = cx^2$

Illustration 24: The solution of differential equation $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ is

Sol: Here by putting y = xv and then integrating both sides we can solve the problem.

Put y = xv $\Rightarrow\ \frac{dy}{dx} = v + x\frac{dv}{dx}$

Hence the given equation becomes $x\frac{dv}{dx} + v = v + \tan v \Rightarrow x\frac{dv}{dx} = \tan v$

$\Rightarrow\ \frac{dv}{\tan v} = \frac{dx}{x}$ $\Rightarrow$ log sinv = logx + logc $\Rightarrow \frac{\sin v}{x} = c \Rightarrow \frac{\sin\left(\frac{y}{x}\right)}{x} = c \Rightarrow cx = \sin\left(\frac{y}{x}\right)$

Illustration 25: Solve $\frac{dy}{dx} = \frac{y^2 - 2xy - x^2}{y^2 + 2xy - x^2}$ given y at x = 1 is –1

Sol: Similar to the problem above, by putting y = vx, we can solve it and then by applying the given condition we will get the value of c.

Let y = vx

$$\Rightarrow \quad v + x\frac{dv}{dx} = \left(\frac{v^2 - 2v - 1}{v^2 + 2v - 1}\right) \qquad \Rightarrow x\frac{dv}{dx} = -\frac{(v^3 + v^2 + v + 1)}{v^2 + 2v - 1}$$

$$\Rightarrow \quad \int \frac{v^2 + 2v - 1}{(v+1)(v^2+1)} dv = c - \log x \Rightarrow \int \frac{2v(v+1) - (v^2+1)}{(v+1)(v^2+1)} dv = c - \log x$$

$$\Rightarrow \quad \log\left[\frac{(v^2+1)x}{v+1}\right] = \log c \quad \Rightarrow \quad \frac{(v^2-1)x}{(v+1)} = c \Rightarrow \frac{x^2+y^2}{y+x} = c$$

$$\Rightarrow \quad k(x^2 + y^2) = x + y$$

Given at x = 1, y = – 1 ⇒ 2k = 0. Hence the required equation is x + y = 0

Illustration 26: Solve $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$ given y at x = 1 is $\sqrt{5}$

Sol: As we know, when $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Hence from given equation $\frac{dy}{dx} = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y}$ so by putting y = vx and integrating both side, we will get the result.

Given $y\left(\frac{dy}{dx}\right) + 2x\frac{dy}{dx} - y = 0$

$$\Rightarrow \quad \frac{dY}{dX} = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-x \pm \sqrt{x^2 + y^2}}{y}$$

Let y = vx

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{\pm\sqrt{v^2+1} - 1}{v} - v \Rightarrow x\frac{dv}{dx} = \frac{\pm\sqrt{v^2+1} - 1 - v^2}{v}$$

$$\Rightarrow \quad \int \frac{v\,dv}{\pm\sqrt{v^2+1} - (1+v^2)} = \log x + C \quad \Rightarrow \quad \int \frac{v\,dv}{\pm\sqrt{v^2+1}\left(\mp\sqrt{v^2+1} + 1\right)} = \log x + C$$

$$\Rightarrow \quad -\ln\left(\mp\sqrt{v^2+1} + 1\right) = \log x + C \qquad \Rightarrow \quad x\left(\mp\sqrt{v^2+1} + 1\right) = c$$

Given at x = 1, y = v = $\frac{dy}{dx} = \frac{7X - 3Y}{-3X + 7Y}$ ⇒ $C = \mp\sqrt{6} + 1$

$$\Rightarrow \quad \mp\sqrt{y^2 + x^2} + x = \mp\sqrt{6} + 1$$

This is the required equation.

Note: The obtained solution has 4 equations.

7.6 Differential Equations Reducible to Homogenous Form

A differential equation of the form $\frac{dv}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c}$, where $\frac{a_1}{\theta_2}\neq\frac{b_1}{b_2}$ can be reduced to homogeneous form by adopting the following procedure

Put x = X + h, y = Y + k, so that $\frac{dy}{dx}=\frac{dy}{dx}$

The equation then transforms to $\frac{dY}{dX}=\frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)}$

Now choose h and k such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$. Then for these values of h and k the equation becomes

$$\frac{dy}{dx}=\frac{a_1X+b_1Y}{a_2X+b_2Y}$$

This is a homogeneous equation which can be solved by putting Y = vX and then Y and X should be replaced by y – k and x – h.

Special case: If $\frac{dy}{dx}=\frac{ax+by+c}{a'x+b'y+c'}$ and $\frac{a}{a'}=\frac{b}{b'}$ = m say, i.e. when coefficient of x and y in numerator and denominator are proportional, then the above equation cannot be solved by the method discussed before because the values of h and k given by the equation will be indeterminate. In order to solve such equations, we proceed as explained in the following example.

Illustration 27: Solve $\frac{dy}{dx}=\frac{3x-6y+7}{x-2y+4}$

Sol: Here the coefficient of x and y in the numerator and denominator are proportional hence by taking 3 common from 3x – 6y and putting x – 2y = v and after that by integrating we will get the result.

$\frac{dy}{dx}=\frac{3x-6y+7}{x-2y+4}=\frac{3(x-2y)+7}{x-2y+4}$; Put x – 2y = v $\Rightarrow 1-2\frac{dy}{dx}=\frac{dy}{dx}$

Now differential equations reduces to $1-\frac{dv}{dx}=2\left(\frac{3v+7}{v+4}\right)$

$\Rightarrow \frac{dv}{dx}=-5\left(\frac{v+2}{v+4}\right)$ $\qquad \Rightarrow \int\left(1+\frac{2}{v+2}\right)dv=-5\int dx$

$\Rightarrow v+2\log|v+2|=-5x+c$ $\qquad \Rightarrow 3x-y+\log|x-2y+2|=c$

Illustration 28: Solution of differential equation (3y – 7x + 7)dx + (7y – 3x + 3) dy = 0 is

Sol: By substituting x = X + h, y = Y + k where (h, k) will satisfy the equation 3y – 7x + 7 = 0 and 7y – 3x + 3 =0 we can reduce the equation and after that by putting Y = VX and integrating we will get required general equation.

The given differential equation is $\frac{dy}{dx}=\frac{7x-3y-7}{-3x+7y+3}$

Substituting x = X + h, y = Y + k, we obtain

$$\frac{dY}{dX}=\frac{(7X-3Y)+(7h-3k-7)}{(-3X+7Y)+(-3h+7k+3)} \quad \text{... (i)}$$

Choose h and k such that 7h – 3k – 7 = 0 and –3h + 7k + 3 = 0.

This gives h = 1 and k = 0. Under the above transformations, equation (i) can be written as

Let Y = VX so that $\frac{dY}{dX} = V + X\frac{dV}{dX}$, we get $\frac{dY}{dX} = \frac{7X - 3Y}{-3X + 7Y}$

$V + X\frac{dV}{dX} = \frac{-3V + 7}{7V - 3} \Rightarrow X\frac{dV}{dX} = \frac{7 - 7V^2}{7V - 3} \qquad \Rightarrow \quad -7\frac{dX}{X} = \frac{7}{2}\cdot\frac{2V}{V^2 - 1}dV - \frac{3}{V^2 - 1}dV$

Integrating, we get

$-7\log X = \frac{7}{2}\log(V^2 - 1) - \frac{3}{2}\log\frac{V - 1}{V + 1} - \log C \qquad \Rightarrow \quad C = (V + 1)^5 (V - 1)^2 X^7 \qquad \Rightarrow \quad C = (y + x - 1)^5 (y - x + 1)^2$

Which is the required solution.

7.7 Linear Differential Equation

A differential equation is linear if the dependent variable (y) and its derivative appear only in the first degree. The general form of a linear differential equation of the first order is

$\frac{dy}{dx} + Py = Q \qquad$ ..... (i)

where P and Q are either constants or functions of x.

This type of differential equation can be solved when they are multiplied by a factor, which is called integrating factor.

Multiplying both sides of (i) by $e^{\int pdx}$, we get $e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int pdx}$

On integrating both sides with respect to x, we get

$ye^{\int Pdx} = \int Qe^{\int Pdx} + c$ which is the required solution, where c is the constant and $e^{\int pdx}$ is called the integrating factor.

Illustration 29: Solve the following differential equation: $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x}$

Sol: We can write the given equation as $e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x}$. By putting $e^{-y} = t$, we can reduce the equation in the form of $\frac{dt}{dx} + Pt = Q$ hence by using integration factor we can solve the problem above.

We have, $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x} \Rightarrow e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x}$ ... (i)

Put $e^{-y} = t$. so that $\frac{dy}{dx}$ in equation (i), we get $-\frac{dt}{dx} + \frac{t}{x} = \frac{1}{x} \Rightarrow \frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x}$ ... (ii)

This is a linear differential equation in t.

Here, $P = -\frac{1}{x}$ and $Q = -\frac{1}{x}$ $\therefore$ I.F. = $e^{\int Pdx} = e^{\int\left(-\frac{1}{x}\right)dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

$\therefore$ The solution of (ii) is, t.(I.F.) = $\frac{dy}{dx} = \frac{3x - 6y + 7}{x - 2y + 4} = \frac{3(x - 2y) + 7}{x - 2y + 4}$

$$t\frac{1}{x} = \int\frac{1}{x}\left(-\frac{1}{x}\right)dx + C \Rightarrow \frac{t}{x} = \frac{1}{x} + c \Rightarrow \frac{e^{-y}}{x} = \frac{1}{x} + C$$

Illustration 30: The function y(x) satisfy the equation $y(x) + 2x\int_0^x \frac{y(x)}{1+x^2}dx = 3x^2 + 2x + 1$. Prove that the substitution $z(x) = \int_0^x \frac{y(x)}{1+x^2}dx$ converts the equation into a first order linear differential equation in z(x) and solve the original equation for y(x)

Sol: By putting $z'(x) = \frac{y(x)}{1+x^2}$ we will get the linear differential equation in z form and then by applying integrating factor we get the result.

Let $z'(x) = \frac{d(x)}{1+x^2}$ $\Rightarrow$ $z'(x) \times (1 + x^2) + 2x(z(x)) = 3x^2 + 2x + 1$

$$\Rightarrow \frac{dz}{dx} + \frac{2x}{1+x^2}z = \frac{3x^2+2x+1}{x^2+1} \quad \text{... (i)}$$

This is a first order linear differential equation in z.

$\therefore$ I.F. $= e^{\int Pdx} = e^{\int \frac{2x}{1+x^2}dx} = 1+x^2$ $\therefore$ Solution of (i) is z(I.F.) $= \int (Q \times I.F)dx + c$

$\Rightarrow z(1 + x^2) = \int \frac{x^3+x^2+x}{x^2+1}(x^2+1)dx + C$ $\Rightarrow z(1 + x^2) = \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + C$ and $y = 3x^2 + 2x + 1 - 2xz$

Illustration 31: Solve the differential equation ysin2x.dx – (1 + y² + cos2x)dy = 0

Sol: Similar to illustration 28, by putting –cos 2x = t, we can reduce the equation in the form of $\frac{dt}{dx}$ + Pt = Q hence by using integration factor we can solve the problem given above.

We have, $y\sin 2x.dx - (1 + y^2 + \cos 2x)dy = 0$

$$\Rightarrow \sin 2x.\frac{dx}{dy} - \frac{\cos 2x}{y} = \frac{1+y^2}{y} \quad \text{... (i)}$$

Putting $-\cos 2x = t$ so that $2\sin 2x\frac{dx}{dy} = \frac{dt}{dy}$ in equation (i), we get $\frac{dt}{dy} + \frac{2}{y}t = 2\left(\frac{1+y^2}{y}\right)$

Here, $P = \frac{2}{y}$ and $Q = 2\frac{1+y^2}{y}$

$\therefore$ I.F. $= e^{\int Pdy} = e^{\int \frac{2}{y}dy} = y^2$ $\therefore$ The solution is t.(I.F.) $= \int (Q \times I.F.)dy + C$

$\Rightarrow$ $t.y^2 = 2\int \frac{1+y^2}{y}.y^2dy = 2\int y + y^3\, dy$ $\Rightarrow$ $t.y^2 = y^2 + \frac{y^4}{2} + C$

On putting the value of t, we get $-.\cos 2x = 1 + \frac{y^2}{2} + Cy^{-2}$

Illustration 32: Solve $y\log y\frac{dx}{dy} + x - \log y = 0$

Sol: By reducing the given equation in the form of $\frac{dx}{dy} + Px = Q$ we can solve this as similar to above illustrations.

We have, $y\log y\frac{dx}{dy} + x - \log y = 0 \Rightarrow \frac{dx}{dy} + \frac{x}{y\log y} = \frac{1}{y}$

This is a linear differential equation in x.

Here $P = \frac{1}{y\log y}$, $Q = \frac{1}{y}$; I.F. $= e^{\int \frac{1}{y\log y}dy} = e^{\log(\log y)} = \log y$